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Chapter 3

Order two differential equations with Galois group $\mathbb{G}_m$

3.1 Introduction

In this chapter, C is an algebraically closed field of characteristic 0, and $K = C(x)$ is the differential field of rational functions with the usual derivation. We are mainly interested in second order homogeneous linear differential equations

$$y'' = r\,y, \quad \text{with } r \in C[x, x^{-1}], \tag{3.1}$$

whose differential Galois group is $\mathbb{G}_m$. Such an equation has a basis of Liouvillian solutions whose logarithmic derivatives are rational functions. Equivalently, the associated Riccati equation

$$u' + u^2 = r, \quad \text{with } r \in C[x, x^{-1}], \tag{3.2}$$

has exactly two rational solutions ([57, 53]). Recall that solutions of the Riccati equation are the logarithmic derivatives $u = y'/y$ of solutions of the differential equation (3.1).

In geometrical terms, the differential equation (3.1) has at most two singular points on $\mathbb{P}^1$, namely zero ($x = 0$) and infinity. We do not consider the case when both these points are *regular singular* (i.e., when $r = c/x^2$ for some $c \in C$, see chapter 1 for the geometrical terminology), since this case is rather trivial, see [57]. Then at least one of these points must be irregular singular. We *assume* that infinity is an irregular singular point. There is no loss of generality here, because the automorphism $x \mapsto x^{-1}$ of $\mathbb{P}^1$ exchanges zero and infinity, so the normalized pull-back of (3.1) with respect to it has the same form (3.1) with the local properties at zero and infinity completely interchanged.

The first result in this chapter (theorem 3.2.1) classifies the local differential Galois groups of (3.1) at zero and infinity. The local Galois group[1] at infinity must be $\mathbb{G}_m$, because there is no much choice at an irregular singular point. Quite surprisingly, it

[1] In the sequel, by a Galois group we always mean a differential Galois group in this chapter.

turns out that the other possible singular point $x = 0$ must be regular singular, and the local Galois group there can be only trivial or $\mathbb{Z}/2\mathbb{Z}$. Additionally, the two global rational solutions of the Riccati equation must have the same residue at zero, which yields some interesting properties of the corresponding Liouvillian solutions of (3.1).

The main aim of this chapter to describe families of differential equations (3.1) whose global Galois group is $\mathbb{G}_\mathrm{m}$. Using the normalized pull-backs of (3.1) with respect to the automorphisms $x \mapsto \beta x$ of $\mathbb{P}^1$ (which fix zero and infinity) one easily obtains one-dimensional families of equations (3.1) with the Galois group $\mathbb{G}_\mathrm{m}$. The main result in this chapter is that up to these transformations there are only discrete families of differential equations (3.1) with Galois group $\mathbb{G}_\mathrm{m}$.

At the end of this chapter we consider equations $y'' = r\,y$ with (more generally) $r \in C(x)$ and the global Galois group being $\mathbb{G}_\mathrm{m}$. It turns out that up to a few simple exceptional situations, such a differential equation must have singular points where the local Galois group is trivial or $\mathbb{Z}/2\mathbb{Z}$. The Liouvillian solutions at these singular points have the same "interesting" properties.

This chapter is organized as follows. In the next section we prove the mentioned result on possible local Galois groups of (3.1) at zero and infinity, and describe the form of the two global solutions of the Riccati equation (3.2). Thereafter we derive a few useful consequences. In section 3.3 we consider families of equations (3.1) with the global Galois group $\mathbb{G}_\mathrm{m}$. After fixing some discrete parameters (such as the order of r at infinity, the number of simple poles of the two global solutions of the Riccati equation) we obtain "universal" families of differential equations with these properties, given by algebraic varieties representing some "natural" functors for this kind of moduli problem, like in chapter 2. In the last section we consider differential equations $y'' = r\,y$ with $r \in C(x)$ (and the Galois group $\mathbb{G}_\mathrm{m}$). The results are illustrated by many explicit examples.

Before going further we explicitly consider some pull-backs which are used in this section. These are pull-backs with respect the finite morphisms $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ of form $x \mapsto t^k$ for some positive integer k. Such a "cyclic" morphism is ramified only above zero and infinity, and the (normalized) pull-back of (3.1) with respect to it is a differential equation over $C(t)$ with the same possible singular points. More specifically, the corresponding field homomorphism $\phi : C(x) \to C(t)$ is defined by $\phi(x) = t^k$, and the normalized pull-back of (3.1) is

$$Y'' = \left(k^2\, t^{2k-2}\, \phi(r) + \frac{k^2 - 1}{4\,t^2} \right) Y. \tag{3.3}$$

This is a differential equation over $C(t)$ with its standard derivation. The solutions of this equation are precisely $t^{(1-k)/2}\,\phi(y)$, where y is a solution of (3.1). If u is a rational solution of the Riccati equation (3.2), then $\tilde{u} = k\,t^{k-1}\,\phi(u) + (1-k)/(2t)$ is a solution of the Riccati equation associated to (3.3). Using this one can prove that if the global Galois group of equation (3.1) is $\mathbb{G}_\mathrm{m}$, then the Galois group of (3.3) is $\mathbb{G}_\mathrm{m}$ as well.

3.2 Local solutions of the Riccati equation

Theorem 3.2.1 *Suppose that the global Galois group of the differential equation (3.1) is $\mathbb{G}_{\mathrm{m}}$, and infinity is an irregular singular point. Let G_0 and G_∞ be the local Galois groups of (3.1) at zero and infinity respectively. Then:*

(i) $\mathrm{ord}_\infty(r) = -2n \le 0$ *with $n \in \mathbb{Z}$, and $G_\infty \cong \mathbb{G}_{\mathrm{m}}$;*

(ii) The two rational solutions of the Riccati equation (3.2) have form

$$u_1 = v + \frac{s}{x} + \frac{F_1'}{F_1}, \qquad u_2 = -v + \frac{s}{x} + \frac{F_2'}{F_2}, \tag{3.4}$$

where

- *$v \in C[x]$ and has degree n;*
- *F_i is a polynomial in $C[x]$ of degree $d_i \ge 0$, has simple roots, and satisfies*[2] *$F_i(0) = 1$;*
- *$s = -(n + d_1 + d_2)/2$;*

(iii) G_0 is a trivial group if $s \in \mathbb{Z}$, and $G_0 \cong \mathbb{Z}/2\mathbb{Z}$ otherwise.

Proof. The local Galois groups G_∞ and $\mathbb{G}_0$ are subgroups of $\mathbb{G}_{\mathrm{m}}$. Since infinity is an irregular singular point, from proposition 6.2 in [57] we conclude that $\mathrm{ord}_\infty(r) = -2n$ with $n \ge 0$, and $G_\infty = \mathbb{G}_{\mathrm{m}}$, as it is claimed by the first statement. From proposition 6.1 in [57] it follows that $\mathrm{ord}_0(r) = -2m$ with $m > 0$ (if zero is a singular point), or $\mathrm{ord}_0(r) \ge 0$.

Let u be a rational solution of the Riccati equation $u' + u^2 = r$. Then $\mathrm{ord}_\infty(u) = -n$ by local computations: if $\mathrm{ord}_0(u) \ge 0$ then we would have $\mathrm{ord}_0(r) \ge 0$; if $\mathrm{ord}_0(u) = -1$ then $\mathrm{ord}_0(r) \ge -2$; if $\mathrm{ord}_0(u) < -1$ then $\mathrm{ord}_0(r) = 2\mathrm{ord}_0(u)$, etc. Analogously, if zero is a singular point, then $\mathrm{ord}_0(r) = -m$. If $\alpha \in C$ is a pole of u, but not a singular point of (3.1), then $\mathrm{ord}_\alpha(u) = -1$ and $u = \frac{1}{x-\alpha} + \dots$ in $C((x-\alpha))$ by similar local computations.

It follows that the two rational solutions of the Riccati equation (3.2) can be written as

$$u_1 = v_1 + \frac{a_1}{x} + w_1 + \frac{F_1'}{F_1}, \qquad u_2 = v_2 + \frac{a_2}{x} + w_2 + \frac{F_2'}{F_2}, \tag{3.5}$$

where for $i = 1, 2$ we have $v_i \in C[x]$, $a_i \in C$, $w_i \in x^{-2}C[x^{-1}]$, and $F_i \in C[x]$ represents poles of u_i other than zero and infinity, thus roots of F_i are simple and we can choose $F_i(0) \ne 0$.

For $i = 1, 2$ let y_i be a solution of $y_i' = u_i y_i$. Elements y_1, y_2 of the Picard-Vessiot extension of $C(x)$ form a basis of the solution space of (3.1). In this basis $\mathbb{G}_{\mathrm{m}}$ acts on the solution space by the matrices $\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in C^*\}$. In particular, the function $f = y_1 y_2$ is invariant under the action of $\mathbb{G}_{\mathrm{m}}$, hence $f \in C(x)$. Then the logarithmic derivative f'/f has only poles of order 1 with integer residues. But

$$\frac{f'}{f} = u_1 + u_2 = (w_1 + w_2) + \frac{a_1 + a_2}{x} + (v_1 + v_2) + \frac{F_1'}{F_1} + \frac{F_2'}{F_2}.$$

[2]Any constant multiples of F_i's give the same solutions (3.4) of the Riccati equation. Usually F_i's are normalized so that they monic, as in the previous chapter, or in [57, 61]. For us the normalization $F_i(0) = 1$ will be more convenient.

We conclude that $w_1 + w_2 = 0$, $v_1 + v_2 = 0$ and $a_1 + a_2 \in \mathbb{Z}$.

Further, $f = y_1 y_2$ satisfies the equation $f''' - 4rf' - 2r'f = 0$, which is the second symmetric power of (3.1), see [57]. Considering this equation locally at infinity we see that $\mathrm{ord}_\infty(f''') > \mathrm{ord}_\infty(4rf')$ because $\mathrm{ord}_\infty(r) \leq 0$, hence the lowest order terms of $4rf'$ and $2r'f$ must cancel. From here one computes that $\mathrm{ord}_\infty(f) = n$. We conclude that

$$n = \mathrm{Res}_\infty\left(\frac{f'}{f}dx\right) = \mathrm{Res}_\infty\left(\frac{a_1 + a_2}{x}dx + \frac{dF_1}{F_1} + \frac{dF_2}{F_2}\right) = -(a_1 + a_2) - d_1 - d_2. \quad (3.6)$$

Suppose now that $\mathrm{ord}_0(r) = -2m \leq -4$. Considering $f''' - 4rf' - 2r'f = 0$ locally at zero one deduces that $\mathrm{ord}_0(r) = m$. Then $m = \mathrm{Res}_0\left(\frac{f'}{f}dx\right) = a_1 + a_2$, because $F_i(0) \neq 0$ for $i = 1, 2$. We obtain the contradiction $n = -m - d_1 - d_2 < 0$. The conclusion is that $\mathrm{ord}_0(r) \geq -2$.

Let $b_{-2} \in C$ be the coefficient of r (locally at zero) to x^{-2}, possibly $b_{-2} = 0$. The local solutions of the Riccati equation (3.2) in $C((x))$ start with $ax^{-1} + \ldots$, where a satisfies $a^2 - a = b_{-2}$. In particular, constants a_1 and a_2 satisfy this equation. If $a_1 \neq a_2$ then $a_1 + a_2 = 1$, and we obtain the contradiction $n = -1 - d_1 - d_2 < 0$. Hence a_1 and a_2 are equal.

Summarizing, we may set $s = a_1 = a_2$, $w_1 = w_2 = 0$ and $v = v_1 = -v_2$. Besides, $\mathrm{ord}_\infty(u_i) = -n$, hence v has degree n. From (3.6) it follows that $s = -(n + d_1 + d_2)/2$. We have proved the part *(ii)* of the theorem .

We have also enough information to apply proposition 6.1 in [57] and conclude the part *(iii)*. □

This theorem gives us detailed information about global solutions of the Riccati equation when the differential equation (3.1) has Galois group $\mathbb{G}_{\mathrm{m}}$. From it one can easily write a general solution of (3.1) as

$$y = C_1\, x^s\, F_1\, \exp\left(\int v\right) + C_2\, x^s\, F_2 \exp\left(-\int v\right). \quad (3.7)$$

On the other hand, a pair of rational functions of form (3.4) satisfying the same Riccati equation $u' + u^2 = r$ gives us a differential equation (3.1) with Galois group $\mathbb{G}_{\mathrm{m}}$. To construct universal families of such equations the following two lemmas will be useful.

Lemma 3.2.2 *Let u_1 and u_2 be two rational functions of form (3.4), as specified in theorem 3.2.1. They satisfy the same Riccati equation (3.2) if and only if the following equation holds for some constant $c \in C$:*

$$F_1' F_2 - F_1 F_2' + 2vF_1 F_2 = c\, x^{d_1 + d_2 + n}. \quad (3.8)$$

Proof. Suppose that the two functions (3.4) satisfy the Riccati equation. Then substituting each u_i into it gives us:

$$\frac{F_1''}{F_1} + 2\left(v + \frac{s}{x}\right)\frac{F_1'}{F_1} + \left(v' + v^2 + \frac{2s}{x}v + \frac{s^2 - s}{x^2}\right) = r, \quad (3.9)$$

$$\frac{F_2''}{F_2} - 2\left(v - \frac{s}{x}\right)\frac{F_2'}{F_2} + \left(-v' + v^2 - \frac{2s}{x}v + \frac{s^2 - s}{x^2}\right) = r. \quad (3.10)$$

By subtracting these two equations one from each other and multiplying the result by F_1F_2 one obtains

$$F_1''F_2 - F_1F_2'' + 2v(F_1'F_2 + F_1F_2') + \frac{2s}{x}(F_1'F_2 - F_1F_2') + \left(2v' + \frac{4s}{x}v\right) F_1F_2 = 0.$$

This equation can be rewritten in the form:

$$(F_1'F_2 - F_1F_2' + 2vF_1F_2)' + \frac{2s}{x}(F_1'F_2 - F_1F_2' + 2vF_1F_2) = 0. \tag{3.11}$$

Hence the polynomial $F_1'F_2 - F_1F_2' + 2vF_1F_2$ satisfies the differential equation $y' = -2s\,y/x$. The general solution of this equation is $y = cx^{-2s}$. Besides, $s = -(d_1 + d_2 + n)/2$.

On the other hand, if F_1, F_2 and v satisfy (3.8), then (3.11) is satisfied. Hence for both $i = 1, 2$ the expression $u_i' + u_i^2$ is the same rational function r. This r is in $C[x, x^{-1}]$, because polynomials F_1 and F_2 satisfying (3.8) must be coprime. $\square$

To state the next lemma, we define (for $v \in C[x]$) the integral $\int v$ to be the primitive of v without constant term. i.e. $\mathrm{ord}_0(\int v) > 0$. Also recall that for $f \in C(x)$ with $\mathrm{ord}_0(f) > 0$ power series of $\exp(f)$ in $C((x))$ is by definition equal to the power series of $1 + f + f^2/2! + f^3/3! + \ldots$.

Lemma 3.2.3 *Two rational functions u_1 and u_2 of form (3.4) satisfy the same Riccati equation (3.2) if and only if locally at zero*

$$\frac{F_1}{F_2} = \exp\left(-2\int v\right) \quad \mathrm{mod} \quad x^{d_1+d_2+n+1}, \tag{3.12}$$

meaning that as formal power series in $C((x))$, both sides of (3.12) are equal up to order $d_1 + d_2 + n$ in x.

Proof. Dividing (3.8) by F_2^2 and expanding at zero one obtains

$$\left(\frac{F_1}{F_2}\right)' = -2v\frac{F_1}{F_2} \quad \mathrm{mod} \quad x^{d_1+d_2+n}. \tag{3.13}$$

Let W be power series expansion of $\exp(-2\int v)$ at zero. It satisfies $W' = -2v\,W$ in $C((x))$. The equation (3.13) says that F_1/F_2 satisfies the same differential equation up to order $x^{d_1+d_2+n-1}$. Local computations show that this holds if and only if power series of F_1/F_2 in $C((x))$ coincide with the power series W up to order $x^{d_1+d_2+n}$. (Note that we use the normalization $F_i(0) = 1$ here.) $\square$

The implication "only if" of the last lemma can also be proved by observing that both basis solutions (3.7) of (3.1) have the local exponent $s < 0$ at zero. Another local exponent at zero is $1 - s$, and there are local solutions with this local exponent because the local Galois group is finite. Since the power series of both $F_1\exp(\int v)$ and $F_2\exp(-\int v)$ start with $1 + \ldots \in C((x))$, their difference must have the local exponent

$1-2s = n+d_1+d_2+1$. In order words, their ratio is equal to 1 in $C((x))/(x^{n+d_1+d_2+1})$, which is equivalent to the statement of theorem 3.2.3.

The formula (3.12) has an important interpretation. In approximation theory ([2]) it is known that for any power series $S \in C((x))$ and any non-negative integers d_1, d_2 there are polynomials G_1 and G_2, of degree at most d_1 and d_2 respectively, such that the power series of the rational function G_1/G_2 approximate S up to order at least $d_1 + d_2$. Moreover, such a rational function G_1/G_2 is unique and is called the *Padé approximation* of S of degree (d_1, d_2). The formula (3.12) says that F_1/F_2 approximates the power series of $\exp(-2\int v)$ in $C((x))$ up to order $n + d_1 + d_2$, so in particular it is the Padé approximation of the power series of $\exp(-2\int v)$ of degree (d_1, d_2). If $n > 0$ then F_1/F_2 approximates $\exp(-2\int v)$ even better than it is ensured by the theory of Padé approximations. For us an important fact is that, according to the next lemma, the approximation relation (3.12) cannot be satisfied if the degree of polynomials F_1, F_2 (or v) is smaller.

Lemma 3.2.4 *Let v be a non-zero polynomial of degree at most n, and let F_1, F_2 be polynomials in $C[x]$ of degree at most d_1 and d_2 respectively. Suppose that formula (3.12) holds. Then $\deg v = n$, $\deg F_1 = d_1$ and $\deg F_2 = d_2$.*

Proof. If formula (3.12) holds, then by differentiating it one obtains

$$\left(\frac{F_1}{F_2}\right)' = -2v \exp\left(-2\int v\right) \mod x^{d_1+d_2+n}.$$

Applying the formula (3.12) itself one derives that formula (3.13) holds. By a simple manipulation one obtains

$$F_1'F_2 - F_1F_2' = -2vF_1F_2 \mod x^{d_1+d_2+n}.$$

If $\deg F_i < d_i$ for $i = 1$ or 2, or $\deg v < n$, then according to this expression the polynomials $F_1'F_2 - F_1F_2'$ and $-2vF_1F_2$ must be equal. But $\deg(-2vF_1F_2) - \deg(F_1'F_2 - F_1F_2') \geq 1 + \deg v$ for $v \neq 0$. This contradiction proves the claim. □

Example 3.1 Here we consider differential equations (3.1) with $n = 0$. Because of the pull-back transformations with respect to the automorphisms $x \mapsto \beta x$ of $\mathbb{P}^1$ we may take $v = -1/2$ without loss of generality. Then in equation (3.12) one can (and must) take F_1/F_2 to be the Padé approximation of $\exp(x)$ of degree (d_1, d_2). From theorem (3.2.3) it follows that

$$u_1 = -\frac{1}{2} - \frac{d_1+d_2}{2x} + \frac{F_1'}{F_1}, \qquad u_2 = \frac{1}{2} - \frac{d_1+d_2}{2x} + \frac{F_2'}{F_2}$$

satisfy the same Riccati equation 3.2. The conclusion is that for any non-negative integers d_1, d_2 there is a differential equation (3.1) with the global Galois group $\mathbb{G}_m$, and such that in the expression (3.4) of the rational solutions of its Riccati equation the polynomials F_1 and F_2 have degree d_1 and d_2 respectively, and $v \in C$. Moreover, up to (normalized) pull-backs with respect to the automorphisms $x \mapsto \beta x$ such an equation is unique.

The rational function r in (3.1) or (3.2) has form $b_{-2}x^{-2}+b_{-1}x^{-1}+b_0$. It can be found without knowing F_1 and F_2 by small local computations at zero and infinity (namely, substituting "locally" the rational functions u_1, u_2 above into the Riccati equation). The explicit differential equation is:

$$y'' = \left(\frac{1}{4} + \frac{d_2 - d_1}{2\,x} + \frac{(d_1+d_2)(d_1+d_2+2)}{4\,x^2}\right) y. \tag{3.14}$$

Let $D = d_1 + d_2$. Then one can check that $x^{-D/2}\,F_1\,e^{-x/2}$ and $x^{-D/2}\,F_2\,e^{x/2}$ with

$$F_1 = \sum_{k=0}^{d_1} \binom{D-k}{d_2} \frac{x^k}{k!}, \qquad F_2 = \sum_{k=0}^{d_2} (-1)^k \binom{D-k}{d_1} \frac{x^k}{k!} \tag{3.15}$$

satisfy equation (3.1) with this r. In this way we have found the Padé approximations of $\exp(x)$ at $x = 0$.

In the easiest case $d_2 = 0$ we have $F_2 = 1$,

$$F_1 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{d_1}}{d_1!} \quad \text{and} \quad r = \frac{1}{4} - \frac{d_1}{2\,x} + \frac{d_1(d_1+2)}{4\,x^2}.$$

3.3 The families of differential equations

Our next objective is to describe the set of differential equations (3.1) with differential Galois group $\mathbb{G}_m$, assuming that infinity is an irregular singular point. This amounts to describing tuples (v, F_1, F_2, r) of polynomials in $C[x]$ or $C[x, x^{-1}]$ such that rational functions (3.4) satisfy the Riccati equation (3.2) with the given r; as in theorem 3.2.1, s is determined by the degree of v and F_i's. We fix non-negative integers n, d_1 and d_2, and consider the set of these tuples such that the degree of v and F_i's is fixed by these integers. We represent this set as an algebraic scheme, like we did for sets of differential equations with one singular point and Liouvillian solutions. As in the previous chapter [61], let $\mathcal{R}_C$ denote the category of commutative C-algebras with 1. Consider the functor $\mathcal{S}_{n,d_1,d_2}$ from $\mathcal{R}_C$ to the category of sets, which associates to a ring $R \in \mathcal{R}_C$ the set of tuples (v, F_1, F_2, r), where

- v is a polynomial in $R[x]$ of degree n, with invertible leading coefficient;[3]
- F_i is a polynomial in $R[x]$ of degree d_i, with $F_i(0) = 1$;
- r is a polynomial in $R[x, x^{-1}]$,

such that, with $s = -(n + d_1 + d_2)/2$, we have

$$F_1'' + 2\left(v + \frac{s}{x}\right) F_1' + \left(-r + v' + v^2 + \frac{2s}{x} v + \frac{s^2 - s}{x^2}\right) F_1 = 0, \tag{3.16}$$

$$F_2'' - 2\left(v - \frac{s}{x}\right) F_2' + \left(-r - v' + v^2 - \frac{2s}{x} v + \frac{s^2 - s}{x^2}\right) F_2 = 0. \tag{3.17}$$

These two equations are equivalent to the requirement that the rational expressions $u_1 = sx^{-1} + v + F_1'/F_1$ and $u_2 = sx^{-1} - v + F_2'/F_2$ in $R[x, x^{-1}, F_1^{-1}, F_2^{-1}]$ satisfy the

[3]The indeterminate x is supposed to be transcendental over R.

Riccati equation $u' + u^2 = r$, compare them with equations (3.9-3.10) in the proof of lemma 3.2.2.

Note that these conditions imply that if $(v, F_1, F_2, r) \in \mathcal{S}_{n,d_1,d_2}(R)$ for some $R \in \mathcal{R}_C$, then F has invertible leading coefficient and invertible discriminant, because otherwise by a specialization $R \to C$, which sends the leading coefficient or the discriminant to zero, one would get a contradiction to theorem 3.2.1 or the remark after lemma 3.2.2. It also automatically follows that the highest degree term in x of r is $b_{2n}x^{2n}$ with b_{2n} invertible in R.

The functor $\mathcal{S}_{n,d_1,d_2}$ is represented as follows (compare this with the previous chapter [61]). Let R_u to be the C-algebra

$$R_u = C[b_{2n}, b_{2n-1}, \ldots, b_0, b_{-1}, b_{-2}, a_n, \ldots, a_0, a_n^{-1}, p_1, \ldots, p_{d_1}, q_1, \ldots, q_{d_2}], \tag{3.18}$$

and consider polynomials $v = a_n x^n + \ldots + a_0$, $F_1 = 1 + p_1 x + \ldots + p_{d_1} x^{d_1}$, $F_2 = 1 + p_2 x + \ldots + p_{d_2} x^{d_2}$ in $R_u[x]$, and

$$r = b_{2n}x^{2n} + \ldots + b_1 x + b_0 + b_{-1}x + b_{-2}x^2 \in R_u[x, x^{-1}]. \tag{3.19}$$

We substitute these polynomials into (3.16) and (3.17), and let T_1 and T_2 be polynomials in $R_u[x, x^{-1}]$, which we obtain after expanding left-hand sides of these equations. Let Q be the ideal in R_u generated by the coefficients of T_1 and T_2. Then the functor $\mathcal{S}_{n,d_1,d_2}$ is represented straightaway by the ring $R_{n,d_1,d_2} = R_u/Q$. In explicit terms, let $\bar{v}$, $\bar{F}_1$, $\bar{F}_2$ and $\bar{r}$ be the homomorphic images in $R_{n,d_1,d_2}[x, x^{-1}]$ of these v, F_1, F_2 and r, respectively. Then for any $R \in \mathcal{R}_C$ we have $\mathcal{S}_{n,d_1,d_2}(R) \cong \mathrm{Mor}_C(R_{n,d_1,d_2}, R)$, so that a homomorphism $R_{n,d_1,d_2} \to R$ gives an element of $\mathcal{S}_{n,d_1,d_2}(R)$ by taking a homomorphic image of the *universal* tuple $(\bar{v}, \bar{F}_1, \bar{F}_2, \bar{r})$, and vice versa, a tuple in $\mathcal{S}_{n,d_1,d_2}(R)$ defines a homomorphism in $\mathrm{Mor}_C(R_{n,d_1,d_2}, R)$ by coefficients of its components. The representing ring R_{n,d_1,d_2} and the universal tuple are unique (up to isomorphism or equivalence) by the general nonsense ([40]). It is clear that for $i = 1, 2$ the leading coefficient and the discriminant of the universal $\bar{F}_i$ are invertible.

Having defined the representing ring R_{n,d_1,d_2}, we define the affine scheme $\mathbf{S}_{n,d_1,d_2} = \mathrm{Spec}\ R_{n,d_1,d_2}$. Then $\mathcal{S}_{n,d_1,d_2}(C)$ is the set of closed points in $\mathbf{S}_{n,d_1,d_2}$. The main result of this chapter is that the algebraic scheme $\mathbf{S}_{n,d_1,d_2}$ is smooth and has dimension 1. We will prove it in the next section.

As we mentioned, one can apply automorphisms $x \mapsto \beta x$ of $\mathbb{P}^1$ to obtain new equations (3.1) with the same Galois group. More generally, for $R \in \mathcal{R}_C$ and $\beta \in R^*$ one can transform a tuple $(v(x), F_1(x), F_2(x), r(x)) \in \mathcal{S}_{n,d_1,d_2}(R)$ to another element

$$\big(\beta\, v(\beta x),\ F_1(\beta x),\ F_2(\beta x),\ \beta^2\, r(\beta x)\big) \quad \text{of} \quad \mathcal{S}_{n,d_1,d_2}(R).$$

This gives an action of the multiplicative algebraic group $\mathbb{G}_\mathrm{m}$ on our functors $\mathcal{S}_{n,d_1,d_2}$ and schemes $\mathbf{S}_{n,d_1,d_2}$. To avoid some confusion, from here on we denote this algebraic group by $\mathbf{G}_\mathrm{m}$ when we speak about its action on $\mathbf{S}_{n,d_1,d_2}$, and we denote it by $\mathbb{G}_\mathrm{m}$ when we mean the Galois group of the differential equation (3.1). Because the action of $\mathbf{G}_\mathrm{m}$ is not a straightforward substitution, we will denote its elements by $x \rightharpoonup \beta x$. In particular, such an element acts on the coefficients of the components of a tuple in $\mathcal{S}_{n,d_1,d_2}(R)$ as follows (with notations as in (3.19), etc.):

$$b_i \mapsto \beta^{i+2} b_i, \quad a_i \mapsto \beta^{i+1} a_i, \quad p_i \mapsto \beta^i p_i, \quad q_i \mapsto \beta^i q_i. \tag{3.20}$$

According to this action one can give these coefficients weights, for instance, weight $i+2$ to b_i, etc. One can easily check that coefficients of T_1 and T_2 (which generate the ideal Q) are homogeneous with respect to this grading, hence the representing ring $R_{n,d}$ is a graded ring.

Since $\mathbf{G}_{\mathrm{m}}$ is a reductive algebraic group, the geometric quotient $\mathbf{S}_{n,d_1,d_2}/\mathbf{G}_{\mathrm{m}}$ exists. From theorem 3.3.1 it follows that the quotient $\mathbf{S}_{n,d_1,d_2}/\mathbf{G}_{\mathrm{m}}$ is zero-dimensional, i.e. $\mathbf{S}_{n,d_1,d_2}$ consists of finitely many $\mathbf{G}_{\mathrm{m}}$ orbits. A natural question is of how many $\mathbf{G}_{\mathrm{m}}$ orbits $\mathbf{S}_{n,d_1,d_2}$ consists. It is preferable to count $\mathbf{G}_{\mathrm{m}}$ orbits with weights, because the points on some orbits have a non-trivial stabilizer. We will consider several examples, and give a conjecture for the total weight formula in general case.

Also note, that if we have $(v, F_1, F_2, r) \in \mathcal{S}_{n,d_1,d_2}(R)$ for some $R \in \mathcal{R}_C$, then $(-v, F_2, F_1, r)$ is an element of $\mathcal{S}_{n,d_2,d_1}(R)$. Points on our schemes correspond to ordered pairs of rational functions satisfying the same Riccati equation (3.2), rather than differential equations (3.1). Each such an equation is represented twice, in some $\mathbf{S}_{n,d_1,d_2}$ and $\mathbf{S}_{n,d_2,d_1}$, and because of theorem (3.2) precisely twice. In particular, if $d = d_1 = d_2$ then there is an involution acting on $\mathcal{S}_{n,d,d}(R)$ for any R, which permutes tuples with the same r. This involution has no fixed points in $\mathbf{S}_{n,d,d}$, but it may have them in $\mathbf{S}_{n,d,d}/\mathbf{G}_{\mathrm{m}}$.

For more convenient explicit computations one can other sets of equations defining $\mathbf{S}_{n,d_1,d_2}$. Note that theorems 3.2.2 and 3.2.3 easily generalize to any C-algebra R instead of C, because transformations of (3.9, 3.10, 3.13) and local computations in $R((x))$ are essentially the same for any R. One can consider a functor which associates to a C-algebra R the set of triples (v, F_1, F_2) of polynomials in $R[x]$ such that (3.8), or equivalently (3.12), is satisfied for some $c \in R$ or $\tilde{c} \in R^*$. Such a functor is equivalent to $\mathcal{S}_{n,d_1,d_2}$, hence the representing rings are isomorphic by Yoneda lemma ([40]). For example, $\mathbf{S}_{n,d_1,d_2}$ is isomorphic to the spectrum of the ring

$$C[a_0, \ldots, a_n, a_n^{-1}, p_1, \ldots, p_{d_1}, q_1, \ldots, q_{d_2}]/\tilde{Q},$$

where $\tilde{Q}$ is the ideal generated by coefficients of $F_1'F_2 - F_1F_2' + 2vF_1F_2$ to x^i for $i = 0, \ldots, n+d_2+d_1-1$, and v and F_i's are the same polynomials as introduced just before the equation (3.19). In this way we have eliminated the coefficients of r. This set of equations is convenient when d_1 and d_2 are small.

In the rest of this section we prefer to use the generalized version of equation (3.12). For convenience, we replace a_i's by variables $\alpha_i = -2a_i/(i+1)$. Then the universal v is equal to

$$v = -\frac{1}{2}\left(\alpha_0 + 2\alpha_1 x + \ldots + (n+1)\alpha_n x^n\right), \tag{3.21}$$

and $-2\int v = \alpha_0 x + \alpha_1 x^2 + \ldots + \alpha_n x^{n+1}$. Let R_v be the ring $C[\alpha_0, \ldots, \alpha_n, \alpha_n^{-1}]$, and let $R_v((x)) := R_v[[x]][x^{-1}]$. Let $\Theta = \theta_0 + \theta_1 x + \theta_2 x^2 + \ldots \in R_v((x))$ denote the power series of $\exp(-2\int v)$. Explicitly,

$$\theta_k = \sum_{i_0+2i_1+\ldots+(n+1)i_n=k} \frac{\alpha_0^{i_0}\,\alpha_1^{i_1}\,\cdots\,\alpha_n^{i_n}}{i_0!\,i_1!\,\cdots\,i_n!}. \tag{3.22}$$

We also define $\theta_k = 0$ for $k < 0$.

The equation (3.12) can be rewritten as $F_1 = \Theta\, F_2 \mod x^{n+d_1+d_2+1}$ in $R_v((x))$. From here one derives equations

$$\begin{aligned} p_j &= \theta_j + \theta_{j-1}q_1 + \ldots + \theta_{j-d_2}q_{d_2}, \quad \text{for } j = 1,\ldots,d_1, & (3.23)\\ 0 &= \theta_j + \theta_{j-1}q_1 + \ldots + \theta_{j-d_2}q_{d_2}, \quad \text{for } j = d_1+1,\ldots,n+d_1+d_2. & (3.24) \end{aligned}$$

Equations (3.23) eliminate the variables p_j, which are not present in equations (3.24). It follows that the functor $\mathcal{S}_{n,d_1,d_2}$ is represented by the ring $R_v[q_1,\ldots,q_{d_2}]/J$, where J is generated by the right hand sides of the equations (3.24). We will use these equations to find the dimension of $\mathbf{S}_{n,d_1,d_2}$ and to prove its smoothness.

Note that equations are linear in q_i's. Their elimination gives us equations in α_i's in the form of determinants of certain matrices whose entries are θ_j's. We have noticed in the previous chapter that equation (3.12) requires that F_1/F_2 has to approximate $\exp(-2\int v)$ in $C((x))$ up to higher order than it is generically possible according to the theory of Padé approximations. Hence the restrictions on the coefficients of v are not surprising. We consider these equations in the next subsection in order to represent $\mathcal{S}_{n,d_1,d_2}$ as a quotient of R_v and express the universal polynomials $\bar{F}_i$ in $R_v[x]$.

Example 3.2 We take $n = 1$, $d_1 = 3$ and $d_2 = 0$. From (3.24) we derive the equation $\alpha_1^2 + \alpha_0\alpha_1 + \alpha_0^4/12 = 0$. This gives two $\mathbf{G}_\mathrm{m}$ orbits defined by

$$\frac{\alpha_1}{\alpha_0^2} = -\frac{1}{2} + \frac{1}{\sqrt{6}} \quad \text{and} \quad \frac{\alpha_1}{\alpha_0^2} = -\frac{1}{2} - \frac{1}{\sqrt{6}}.$$

Setting $\alpha_0 = -\sqrt{6}$ and $\alpha_1 = \sqrt{6} - 3$ and using (3.23), $s = -2$ and $r = u_i' + u_i^2$ one gets the following representative of one of these orbits:

$$\begin{aligned} v = (3-\sqrt{6})x + \frac{\sqrt{6}}{2}, \quad F_1 = 1 - \sqrt{6}x + \sqrt{6}x^2 + (2\sqrt{6}-6)x^3, \quad F_2 = 1,\\ r = (15-6\sqrt{6})x^2 + (3\sqrt{6}-6)x + \frac{21-6\sqrt{6}}{2} + \frac{2\sqrt{6}}{x} + \frac{6}{x^2}. \quad (3.25) \end{aligned}$$

Here $s = -2$. A representative of another orbit is obtained by the conjugation $\sqrt{6} \mapsto -\sqrt{6}$.

3.3.1 Dimension and smoothness

In this section we prove the main result of the chapter, and consider a representation of $\mathcal{S}_{n,d_1,d_2}$ as a quotient of the ring R_v.

To prove the smoothness of $\mathbf{S}_{n,d_1,d_2}$ we use the same techniques as in the previous chapter, based on the lemma 2.11 in [49]. For a point $Q \in \mathcal{S}_{n,d_1,d_2}(C)$ let $\mathcal{S}_{n,d_1,d_2}^Q$ be the functor on local Artinian rings $R \in \mathcal{R}_C$ with residue field C, so that $\mathcal{S}^Q_{n,d_1,d_2}(R)$ is the set of elements of $\mathcal{S}_{n,d_1,d_2}(R)$ which are mapped to Q by the induced $\mathcal{S}_{n,d_1,d_2}(R) \to \mathcal{S}_{n,d_1,d_2}(C)$. Let $C[\varepsilon]$ be the ring of dual numbers, $\varepsilon^2 = 0$ as usual. Then the tangent space of $\mathbf{S}_{n,d_1,d_2}$ at a point Q can be identified with $\mathcal{S}^Q_{n,d_1,d_2}(C[\varepsilon])$.

Theorem 3.3.1 *The algebraic scheme $\mathbf{S}_{n,d_1,d_2}$ is smooth, reduced, of dimension 1.*

Proof. First note that $\mathbf{S}_{n,d_1,d_2}$ is defined by $n + d_2$ equations (3.24) in $n + d_2 + 1$ algebraically independent variables. It follows that any reduced irreducible component of $\mathbf{S}_{n,d_1,d_2}$ has dimension at least one. We will show that the dimension of the tangent space of $\mathbf{S}_{n,d_1,d_2}$ at any point is one.

Let Q be a point on $\mathbf{S}_{n,d_1,d_2}$. In this proof, v, F_1, F_2 denote the components of Q as an element (v, F_1, F_2, r) of $\mathcal{S}_{n,d_1,d_2}(C)$. So they are polynomials or rational functions in $C[x]$ or $C(x)$. Respectively, $\Theta = \theta_0 + \theta_1 x + \ldots$ denotes the power series $\exp(-2\int v)$ in $C((x))$, so $\theta_i \in C$.

Let $\tilde{Q}$ be a point in $\mathcal{S}^Q_{n,d_1,d_2}(C[\varepsilon])$. We can write $\tilde{Q} = (v+\varepsilon\tilde{v}, F_1+\varepsilon G_1, F_2+\varepsilon G_2, \ldots)$, where $\tilde{v}$ is a polynomial in $C[x]$ of degree $\le n$, and (for $i = 1, 2$) G_i is a polynomial in $C[x]$ of degree $\le d_i$ with $G_i(0) = 0$. Using the definition of the exponential power series (the remark before lemma 3.2.3) one can check that $\exp(-2\int(v + \varepsilon\tilde{v})) = \Theta + \varepsilon\, w\, \Theta$, where $w = -2\int \tilde{v}$. Recall that $w(0) = 0$ by the definition before the same lemma.

We define here the degree of the zero polynomial to be zero. We claim that if $\deg \tilde{v} < n$ then $\tilde{v} = G_1 = G_2 = 0$. With this assumption $\deg w \le n$. The relation (3.12) for $\tilde{Q}$ in the ring $C[\varepsilon]$ reads

$$F_1 + \varepsilon G_1 = (F_2 + \varepsilon G_2)(\Theta + \varepsilon\, w\, \Theta) \mod x^{n+d_1+d_2+1}.$$

From here one derives

$$G_1 = (G_2 + w\, F_2)\,\Theta \mod x^{n+d_1+d_2+1}. \tag{3.26}$$

If $G_1 = 0$ then $w = 0$ (because otherwise $\deg(w\, F_2) > \deg(G_2)$), and consequently $G_2 = 0$. If $G_1 \neq 0$, let k be the largest integer such that x^k divides G_1. Then x^k also divides $G_2 + w\, F_2$. But equation (3.26) implies that $G_1/(G_2 + w\, F_2)$ approximates Θ up to order $\ge n + d_1 + d_2 - k$, whereas the numerator and denominator of this rational function have degree $\le d_1 - k$ and $\le n + d_2 - k$ respectively. Since $k \ge 1$ we get a contradiction with lemma 3.2.4.

From here it follows that the Krull dimension of the local ring $\mathcal{O}_Q$ is at most one, because if the tangent space $\mathcal{S}^Q_{n,d_1,d_2}(C[\varepsilon])$ has dimension ≥ 1, then the would exist a non-trivial element $\tilde{Q} = (v + \varepsilon\tilde{v}, \ldots)$ such that $\deg \tilde{v} \le n - 1$, contradicting the above claim. Since $\mathcal{O}_Q$ is a quotient of $n + d_2 + 1$-dimensional ring by $n + d_2$ elements (see equations (3.24)), it follows that $\mathcal{O}_Q$ is a regular local ring of dimension 1 for any $\tilde{Q} \in \mathbf{S}_{n,d_1,d_2}$. The statement of the theorem follows. $\Box$

Here we introduce a representation of $\mathcal{S}_{n,d_1,d_2}$ by a quotient of R_v and express the universal polynomials F_i in it. This representation will be used for $n = 1$. Let $K =$

$-2s = n + d_1 + d_2$, and consider the matrix

$$M = \begin{pmatrix} \theta_{d_1+1} & \theta_{d_1} & \theta_{d_1-1} & \dots & \theta_{d_1-d_2+1} \\ \theta_{d_1+2} & \theta_{d_1+1} & \theta_{d_1} & \dots & \theta_{d_1-d_2+2} \\ \theta_{d_1+3} & \theta_{d_1+2} & \theta_{d_1+1} & \dots & \theta_{d_1-d_2+3} \\ \dots & \dots & \dots & \dots & \dots \\ \theta_K & \theta_{K-1} & \theta_{K-2} & \dots & \theta_{K-d_2} \end{pmatrix}. \tag{3.27}$$

Let $\tilde{J}$ be the ideal in R_v generated by $(d_2+1)\times(d_2+1)$ minors of M. Then the functor $\mathcal{S}_{n,d_1,d_2}$ is represented by the ring $R_v/\tilde{J}$. Indeed, any d_2+1 equations from (3.24) must have a solution in the representing ring. They are linear in q_i's, and since the associated matrix is a submatrix of M of maximal size, all $(d_2+1)\times(d_2+1)$ minors must vanish. It remains to prove the existence of the universal polynomials v and F_i's. First consider the homogeneous part of (3.24)

$$\theta_{j-1}q_1 + \ldots + \theta_{j-d_2}q_{d_2} = 0, \quad \text{for} \quad j = d_1+1, \ldots, n+d_1+d_2. \tag{3.28}$$

For any homomorphism $R_v \to C$ this system with specialized θ_j's has only trivial solutions in C, because otherwise we would get a contradiction to lemma 3.2.4 (namely, a solution of (3.28) gives a polynomial $\tilde{F}_2$ of degree less than d_2 such that $\tilde{F}_2\,\Theta$ is a polynomial of degree at most d_1). This implies that $d_2 \times d_2$ minors formed by the last d_2 columns of M generate the unit ideal in R_v. Consider M as a matrix over R_v/J, we can apply lemma 2.3.6 from the previous chapter, which yields that the first column of M is a linear combination of the rest columns. Hence a solution of (3.24) over R_v/J exists. This gives us the universal polynomial F_2, and due to equations (3.23) we also get F_1. The universal v is given by (3.21), of course.

Using linear algebra one can derive explicit expressions for universal F_1 and F_2. Considering the first d_2 equations in (3.24), and applying Cramer's rule and the standard expansion formulas of determinants, one concludes that R_v/J-multiples of F_1 and F_2 are, respectively,

$$\sum_{j=0}^{d_1} \det \begin{pmatrix} \theta_j & \theta_{j-1} & \theta_{j-2} & \dots & \theta_{j-d_2} \\ \theta_{d_1+1} & \theta_{d_1} & \theta_{d_1-1} & \dots & \theta_{d_1-d_2+1} \\ \theta_{d_1+2} & \theta_{d_1+1} & \theta_{d_1} & \dots & \theta_{d_1-d_2+2} \\ \dots & \dots & \dots & \dots & \dots \\ \theta_{d_1+d_2} & \theta_{d_1+d_2-1} & \theta_{d_1+d_2-2} & \dots & \theta_{d_1} \end{pmatrix} x^j \tag{3.29}$$

$$\text{and} \qquad \det \begin{pmatrix} 1 & x & x^2 & \dots & x^{d_2} \\ \theta_{d_1+1} & \theta_{d_1} & \theta_{d_1-1} & \dots & \theta_{d_1-d_2+1} \\ \theta_{d_1+2} & \theta_{d_1+1} & \theta_{d_1} & \dots & \theta_{d_1-d_2+2} \\ \dots & \dots & \dots & \dots & \dots \\ \theta_{d_1+d_2} & \theta_{d_1+d_2-1} & \theta_{d_1+d_2-2} & \dots & \theta_{d_1} \end{pmatrix}. \tag{3.30}$$

Moreover, all $d_2 \times d_2$ minors of M, formed by successive rows and columns, are invertible in R_v/J. Otherwise after a specialization $R_v/J \to C$ we would get a homogeneous system of d_2 successive equations in (3.28) with a solution which would finally contradict the the approximation lemma 3.2.4 (in more detail, the solution would give a polynomial $\tilde{F}_2$ of degree $\le d_2 - 1$ such that $F_1\tilde{F}_2/F_2$ would turn out to be a polynomial of degree $\le d_1 - 1$). Then it follows that F_1 and F_2 are equal up to a constant in $(R_v/J)^*$ to (3.29) and (3.30).

3.3.2 Counting equations

In this subsection we count $\mathbf{G}_{\mathrm{m}}$ orbits in $\mathbf{S}_{n,d_1,d_2}$ for small n or d_i's. The case $n=0$ was in fact considered in an example at the end of section 3.2. One can also use equations (3.23–3.24) to conclude that there is exactly one $\mathbf{G}_{\mathrm{m}}$ orbit in $\mathbf{S}_{0,d_1,d_2}$ for any d_2.

Theorem 3.3.2 $\mathbf{S}_{1,d_1,d_2}$ *consists of* $\lceil \frac{(d_1+1)(d_2+1)}{2} \rceil$ $\mathbf{G}_{\mathrm{m}}$ *orbits.*

Proof. The matrix (3.27) is a square matrix in this case. We get exactly one equation in α_i's, namely $\det M = 0$. This determinant must be homogeneous in R_v, with respect to weights introduced in (3.20). Since the weight of θ_j is j, the determinant of M has weight $(d_1+1)(d_2+1)$. This determinant factors (over C) into factors $a_1 + \zeta a_0$ of weight 2 and (possibly) a factor a_0 of degree 1. All factors are distinct, because $\mathbf{S}_{1,d_1,d_2}$ is smooth. Hence there is at most one factor of degree 1, so there are $\lceil (d_1+1)(d_2+1)/2 \rceil$ factors in total. Because each of them represents a $\mathbf{G}_{\mathrm{m}}$ orbit in $\mathbf{S}_{1,d_1,d_2}$ (and vice versa), the claim of the theorem follows. □

In general, we conjecture the following *weight formula* for the number of $\mathbf{G}_{\mathrm{m}}$ orbits in $\mathbf{S}_{n,d_1,d_2}$. Recall that in a weight formula we count a $\mathbf{G}_{\mathrm{m}}$ orbit whose points are fixed by a subgroup of $\mathbf{G}_{\mathrm{m}}$ of order k with the weight $1/k$. For example, take $n=1$ and consider a $\mathbf{G}_{\mathrm{m}}$ orbit in $\mathbf{S}_{1,d_1,d_2}$ with $a_0 = 0$. Then all variables p_i, q_i and b_i with odd weight zero, because they are multiples of a_0. Hence such an orbit is stabilized by $\mathbb{Z}/2\mathbb{Z}$ and has weight $1/2$. Note that a differential equation represented by a point on a $\mathbf{G}_{\mathrm{m}}$ orbit of weight $1/k$ is a pull-back of a differential in $\mathcal{S}_{(n+1)/k-1,d_1/k,d_2/k}(C)$ with respect to a finite morphism $X \mapsto x^k$, because the only non-zero coefficients are those whose weight is divisible by k.

If the total weight of $\mathbb{G}_{\mathrm{m}}$ orbit in all $\mathbf{S}_{n,d_1,d_2}$'s is known, then one can recursively compute the number of orbits with weight 1, or the total number of orbits counted without weights, because all possible pull-backs come from all $\mathbf{G}_{\mathrm{m}}$ orbits in $\mathbf{S}_{(n+1)/k-1,d_1/k,d_2/k}$ for each k dividing $\gcd(n+1,d_1,d_2)$. Our conjectured total weight of orbits in $\mathbf{S}_{n,d_1,d_2}$ is

$$\frac{1}{n+1}\binom{n+d_1}{n}\binom{n+d_2}{n}. \tag{3.31}$$

This is the correct formula when $n=0$ or $n=1$. It is also supported by the following examples with small n and d_i's.

Example 3.3 $\boxed{d_1 = d_2 = 0.}$ In this case $F_1 = F_2 = 1$, and according to lemma (3.2.2) we have $v = c\,x^n$. We can normalize $v = x^n$. This gives us two rational functions $u_1 = x^n - \frac{n}{2x}$ and $u_2 = -x^n - \frac{n}{2x}$ which satisfy the same Riccati equation. The corresponding differential equation is

$$y'' = \left(x^{2n} + \frac{n(n+2)}{4\,x^2}\right) y. \tag{3.32}$$

It represents the only $\mathbb{G}_{\mathrm{m}}$ orbit in $\mathbf{S}_{n,0,0}$. Note that v is fixed under the subgroup $\mathbb{Z}/(n+1)\mathbb{Z}$ of $\mathbf{G}_{\mathrm{m}}$. Accordingly, the $\mathbb{G}_{\mathrm{m}}$ orbit in $\mathbf{S}_{n,0,0}$ has weight $1/(n+1)$, supporting the conjectured formula (3.31). Also note that the equation (3.32) is the pull-back of

$Y'' = Y$ with respect to $X \mapsto x^{n+1}/(n+1)$.

Example 3.4 $\boxed{d_1 = 1, d_2 = 0.}$ We can take $F_1 = x - 1$ and $F_2 = 1$ to get rid of the $\mathbf{G}_{\rm m}$ action. (We do not have to worry about the normalization of F_i's much.) According to the lemma 3.2.2 we must have $2v = (c\,x^{n+1} - 1)/(x-1)$, which is a polynomial in $C[x]$ if and only if $c = 1$. Hence there is only one possibility, $v = \frac{1}{2}(1 + x + x^2 + \ldots + x^n)$. Therefore for any n there is exactly one $\mathbb{G}_{\rm m}$ orbit in $\mathbf{S}_{n,1,0}$. This orbit has weight 1, because the degree of F_1 has no divisors > 1. Hence formula (3.31) is true in this case.

To compute the corresponding r, one can write down the two rational solutions

$$u_1 = \frac{1}{2}\sum_{k=0}^{n} x^k - \frac{n+1}{2\,x} + \frac{1}{x-1}, \qquad u_2 = -\frac{1}{2}\sum_{k=0}^{n} x^k - \frac{n+1}{2\,x},$$

of the same Riccati equation, and compute

$$\begin{aligned} r &= \frac{1}{4}\sum_{k=0}^{2n}(k+1)\,x^{2n-k} + \frac{n+1}{2\,x} + \frac{(n+1)(n+3)}{4\,x^2} \\ &= \frac{x^{2n+2}-1}{4\,(x-1)^2} - \frac{n+1}{2\,x\,(x-1)} + \frac{(n+1)(n+3)}{4\,x^2}. \end{aligned}$$

As a matter of fact, the general solution of the differential equation $y'' = ry$ with this r is

$$C_1\,(x-1)\,x^{-\frac{n+1}{2}}\,e^{\frac{1}{2}\left(x+\frac{x^2}{2}+\ldots+\frac{x^{n+1}}{n+1}\right)} + C_2\,x^{-\frac{n+1}{2}}\,e^{-\frac{1}{2}\left(x+\frac{x^2}{2}+\ldots+\frac{x^{n+1}}{n+1}\right)}.$$

Example 3.5 $\boxed{d_1 = d_2 = 1.}$ We can take $F_1 = x-1$ and $F_2 = x - \xi$, where $\xi \in C\backslash\{0,1\}$. With this choice we divided out the $\mathbf{G}_{\rm m}$ action. Since the action of $\mathbf{G}_{\rm m}$ does not fix F_i's, to each ξ there corresponds one $\mathbf{G}_{\rm m}$ orbit of weight 1. Note however, that we do not take into account the involution $(v, F_1, F_2, r) \mapsto (-v, F_2, F_1, r)$ on $\mathcal{S}_{n,1,1}(C)$. If $\xi \neq -1$, then the corresponding differential equations will be represented in two different $\mathbf{G}_{\rm m}$ orbits.

The lemma 3.2.2 implies that we must have $2v = (c\,x^{n+2} + \xi - 1)/(x-1)(x-\xi)$. Hence $c = 1 - \xi$ and $c\,\xi^{n+2} = 1 - \xi$. It follows that $\xi^{n+2} = 1$, and since $\xi = 1$ is excluded, we have $n+1$ $\mathbf{G}_{\rm m}$ orbits, as formula (3.31) predicts.

For each ξ the corresponding v is equal to

$$v = \frac{(1-\xi)x^{n+2} - (1-\xi)}{2(x-1)(x-\xi)} = \sum_{k=0}^{n} \frac{1-\xi^{k+1}}{2}\,x^{n-k} = \sum_{k=0}^{n} \frac{1-\xi^{-k-1}}{2}\,x^k.$$

One can write the two rational solutions of the corresponding Riccati equation in a compact form as

$$u_i = \frac{(-1)^{i-1}(1-\xi)\,x^{n+2} + 2\,x - 1 - \xi}{2\,(x-1)(x-\xi)} - \frac{n+2}{2x}, \qquad \text{for} \quad i = 1, 2.$$

From here one can get an expression for r, and the differential equation itself

$$y'' = \left(\frac{(1-\xi)^2 (x^{2n+4}-1)}{4\,(x-1)^2\,(x-\xi)^2} - \frac{(n+1)(2\,x-1-\xi)}{2\,x\,(x-1)(x-\xi)} + \frac{(n+2)(n+4)}{4\,x^2} \right) y. \tag{3.33}$$

Example 3.6 $\boxed{d_1 = 2, d_2 = 0.}$ Similarly, we take $F_1 = (x-1)(x-\xi)$ and $F_2 = 1$, with $\xi \in C \setminus \{0,1\}$. However, some $\mathbf{G}_{\rm m}$ orbits may be represented by two ξ's. Namely, replacing ξ by ξ^{-1} gives a $\mathbf{G}_{\rm m}$-equivalent element of $\mathcal{S}_{n,2,0}$. If $\xi = -1$, the corresponding orbit is fixed by this $\mathbb{Z}/2\mathbb{Z}$, hence it would have weight $1/2$. Other orbits have weight 1.

From lemma 3.2.2 we conclude that $2v = (c\,x^{n+2} - 2x + 1 + \xi)/(x-1)(x-\xi)$. This is a polynomial if $c = 1-\xi$ and $c\,\xi^{n+2} = \xi - 1$. Hence $\xi^{n+2} = -1$. To count the total weight of $\mathbb{G}_{\rm m}$ orbits we may assign weight $1/2$ to each ξ. We obtain the same total weight $(n+2)/2$ as formula (3.31) predicts.

Similarly, we can write the corresponding v as

$$v = \frac{(1-\xi)x^{n+2} - 2x + 1 + \xi}{2(x-1)(x-\xi)} = \sum_{k=0}^{n} \frac{1-\xi^{k+1}}{2}\, x^{n-k} = \sum_{k=0}^{n} \frac{1+\xi^{-k-1}}{2}\, x^k.$$

To express r and rational solutions of the Riccati equation one can use the same formulas as in the previous example.

In particular, we may set $\xi = -1$ in the last two examples. In both cases we get the differential equation

$$y'' = \left(\frac{x^{2n+4} - x^2}{(x^2-1)^2} - \frac{n+1}{x^2-1} + \frac{(n+2)(n+4)}{4\,x^2} \right) y. \tag{3.34}$$

It is represented by a point in $\mathbf{S}_{n,1,1}$ if n is even, and by a point in $\mathbf{S}_{n,2,0}$ if n is odd. In the last case the equation (3.34) is the pull-back of an equation in $\mathcal{S}_{(n-1)/2,1,0}(C)$. As we will see in the next section, (3.34) with even n is the pull-back of an equation with the Galois group $\mathbb{D}_\infty$.

Example 3.7 $\boxed{n = 3, d_1 = 6, d_2 = 0.}$ Here we use the variables $\alpha_0, \ldots, \alpha_3$. Equations (3.24) tells us that $\theta_i = 0$ for $i = 7, 8$ and 9. If $\alpha_0 \neq 0$ then we get rid of $\mathbf{G}_{\rm m}$ action by setting $a_0 = 1$. Using a Gröbner bases computer package one can eliminate α_2 and α_3 from these three equations. The result is a single equation of degree 20 in α_1. This gives us twenty $\mathbf{G}_{\rm m}$ orbits of weight 1. If $\alpha_0 = 0$ one gets equations

$$\begin{aligned} \alpha_2\,(\alpha_1^2 + 2\,\alpha_3) &= 0, \\ \alpha_1^4 + 12\,\alpha_2^1\alpha_3 + 12\,\alpha_1\alpha_2^2 + 12\,\alpha_3^2 &= 0, \\ \alpha_2\,(\alpha_1^3 + \alpha_2^2 + 6\,\alpha_1\alpha_3) &= 0. \end{aligned}$$

By solving these equation one finds two families of solutions defined by $\alpha_2 = 0$, $\alpha_1^4 + 12\,\alpha_2^1\alpha_3 + 12\,\alpha_3^2 = 0$. They represent two $\mathbb{G}_{\rm m}$ orbits of weight $1/2$. The corresponding differential equations are pull-backs of differential equations in $\mathcal{S}_{1,3,0}(C)$. They were considered at the end of section 3.3, see formula (3.25). The correct total weight 21 is predicted by formula (3.31).

3.4 Differential equations with $r \in C(x)$

In this section we consider differential equations

$$y'' = r\,y, \qquad \text{with} \quad r \in C(x). \tag{3.35}$$

with the global Galois group $\mathbb{G}_{\mathrm{m}}$. As in the case $r \in C[x, x^{-1}]$, the associated Riccati equation

$$u' + u^2 = r, \quad \text{with} \quad r \in C(x), \tag{3.36}$$

has exactly two rational solutions ([57, 53]).

We are going to generalize theorem 3.2 and its conclusions. Our main result about equations (3.35) with Galois group $\mathbb{G}_{\mathrm{m}}$ is that apart form a few simple exceptions, such equations have at least one regular singular point with special properties. Namely, the local Galois group at such a point is trivial or $\mathbb{Z}/2\mathbb{Z}$, the two rational solutions of (3.36) have the same residues there, and the two Liouvillian solutions of (3.35) whose logarithmic derivatives are the rational Riccati solutions, have the same power series expansions at it up to some order. These main results follow from the following lemma.

Proposition 3.4.1 *Suppose that the differential equation (*3.35*) has Galois group $\mathbb{G}_{\mathrm{m}}$. Assume that the two global rational solutions u_1 and u_2 of the Riccati equation (*3.36*) have different residues at all regular singular points of (*3.35*). Then $r = 1/P^2$, where P is a polynomial in $C[x]$ of degree ≤ 2.*

Proof. First assume that infinity is a singular point. Since the local Galois group must be a subgroup of $\mathbb{G}_{\mathrm{m}}$, according to the proposition 6.2 in [57] we have $\mathrm{ord}_\infty(r) = -2n_\infty$ with integer $n_\infty \geq -1$. Similarly, if $\alpha \in C$ is any other singular point of (3.35) then $\mathrm{ord}_\alpha(r) = -2n_\alpha$, with integer $n_\alpha \geq 1$.

Let u_1 and u_2 be the two global rational solutions of the Riccati equation. For $i = 1, 2$ let y_i be a solution of $y_i' = u_i y_i$. Similarly as in the theorem 3.2, the product $f = y_1 y_2$ is invariant under the action of $\mathbb{G}_{\mathrm{m}}$, thus $f \in C(x)$. Then $u_1 + u_2 = f'/f$ has only poles of order 1 with integer residues. Besides, f satisfies the equation of the second symmetric power of (3.35):

$$f''' - 4\,r\,f' - 2\,r'\,f = 0. \tag{3.37}$$

We claim that under the assumptions made we have $\mathrm{ord}_\alpha(f) = n_\alpha$ for any singular point $\alpha \in \mathbb{P}^1$ of (3.35). If α is an irregular singular point, then by local considerations we have $\mathrm{ord}_\alpha(f''') > \mathrm{ord}_\alpha(4rf')$, so the lowest order terms of $4rf'$ and $2r'f$ must cancel, and this gives $\mathrm{ord}_\alpha(f) = n_\alpha$. (Compare with the proof of theorem 3.2.) Suppose that α is a regular singular point. If $\alpha = \infty$ then $r = b_{-2}x^{-2} + \ldots \in C((x^{-1}))$, and for $i = 1, 2$ we have locally $u_i = a_i x^{-1} + \ldots$ with a satisfying $a_i^2 - a_i = b_{-2}$. Because $a_1 \neq a_2$ we have $a_1 + a_2 = 1$ and $\mathrm{Res}_\infty((u_1 + u_2)dx) = -1$, therefore $\mathrm{ord}_\alpha(f) = n_\alpha$. If $\alpha \in C$ then $r = b_{-2}(x-\alpha)^{-2} + \ldots \in C((x-\alpha))$, and by similar local computations we conclude that $\mathrm{ord}_\alpha(f) = n_\alpha$. The claim follows.

From this claim we conclude that f may have a pole only at infinity. Indeed, every pole of f is a singular point of (3.35), because otherwise the lowest order term of f''' at α would not be canceled on the left-hand side of (3.37). Then we have $\mathrm{ord}_\alpha(f) = n_\alpha$, but $n_\alpha > 0$ if $\alpha \in C$. Moreover, if infinity is a pole of f, then $\mathrm{ord}_\infty(f) = -1$, hence

$f = c_1\,x + c_0$. From (3.37) one derives $r'/r = -2/(x + c_0/c_1)$ and $r = \tilde{c}/(c_1\,x + c_0)^2$. If infinity is not a pole of f,, then $f \in C$ and by the same (3.37) also $r \in C$. This proves the lemma in the case when infinity is a singular point of (3.35).

If infinity is not a singular point of (3.35), then the differential equation must have a singular point in C. We can move a singularity to infinity by an automorphism of $\mathbb{P}^1$ and use the above results. By the inverse automorphism one transforms the derived equations $r = c$ or $r = c/(x - \alpha_0)^2$ with a singularity at infinity to equations of form $r = \tilde{c}/(x - \alpha_1)^4$ or $r = \tilde{c}/(x - \alpha_1)^2(x - \alpha_2)^2$ respectively. All these equations have form $1/P^2$ with $P \in C[x]$ of degree ≤ 2. $\square$

Corollary 3.4.2 *Suppose that the differential equation (*3.35*) has Galois group $\mathbb{G}_{\rm m}$, and r is not of form $1/P^2$ with a polynomial $P \in C[x]$ of degree ≤ 2. Then there is a regular singular point with the property, that the two global rational solutions of the Riccati equation (*3.36*) have the same residue at α. Moreover, twice this residue is a negative integer, and the local Galois group at α is trivial or $\mathbb{Z}/2\mathbb{Z}$.*

Proof. The proposition 3.4.1 implies that there is a regular singular point α of (3.35) such that the two rational solutions u_1 and u_2 have the same residue s at α. Because $u_1 + u_2$ is the logarithmic derivative of a rational function (see the proof of the proposition), $2s$ must be an integer. The local Galois group at α is trivial if $s \in \mathbb{Z}$, and it is $\mathbb{Z}/2\mathbb{Z}$ otherwise. Besides, s is equal to the smallest local exponent, because there is only one (up to a constant multiple) local solution of (3.35) with the greater local exponent. Because the sum of local exponents is 1, and $s^2 - s \notin \{-3/4, 1\}$ (then the local Galois group would be $\mathbb{G}_{\rm a}$), we conclude that $s < 0$. $\square$

In the following we *assume* that the differential equation (3.35) has Galois group $\mathbb{G}_{\rm m}$, and that $1/r$ is not the square of a polynomial in $C[x]$ of degree ≤ 2. Equivalently, the set of singular points of (3.35) has at least two elements, and it is not a set of two regular singular points. For simplicity, we always choose infinity as a singular point where the residues of the global Riccati equations are supposed to be different. Note that such a singular point must exist, otherwise the Galois group would be smaller.

To write down families of differential equations $y'' = ry$ with Galois group $\mathbb{G}_{\rm m}$, we choose two sets of points on $\mathbb{P}^1$. One set S_0 consists of points which are supposed to be regular singular for the differential equation (3.35), and where the residues of the two global solutions of the Riccati equation must be equal to the same negative half-integer. We prescribe these residues to the points in this set. Another set S_1 consists of the remaining singular points, which are either irregular singular, or the two solutions of the Riccati equation are supposed to have different residues at them. We prescribe the order of the pole of the Riccati solutions u_1 and u_2 at these points. For convenience we assume that infinity is an element of S_1. We also choose two non-negative integers d_1 and d_2. Suppose that we have prescribed the residue $-m_\alpha/2$ to $\alpha \in S_0$, and that we have declared $\mathrm{ord}_\alpha(u_i) = -n_\alpha$ for $\alpha \in S_1$. We have $n_\alpha \geq 1$ if $\alpha \in C$, and $n_\infty \geq -1$. The number of other poles of u_1 and u_2 is supposed to be d_1 and d_2 respectively. The equations (3.35) with Galois group $\mathbb{G}_{\rm m}$ and satisfying the prescribed data will be said to have the type $(\{[\alpha]_{m_\alpha}\}_{\alpha \in S_0}, \{[\alpha]^{n_\alpha}\}_{\alpha \in S_1}, d_1, d_2)$. For example, equations considered in the main part of this chapter have type $([0]_{n+d_1+d_2}, [\infty]^n, d_1, d_2)$. The type defines the simple poles of $u_1 + u_2$ and their residues. Since $u_1 + u_2$ is the logarithmic derivative of a rational function, we must have

$$\sum_{\alpha \in S_0} m_\alpha = \sum_{\alpha \in S_1} n_\alpha + d_1 + d_2. \tag{3.38}$$

Now we would like to consider the set of equations of the same fixed type as an algebraic scheme by representing similar functors as in the previous sections. In naive terms, let $v_\infty = 0$ if $n_\infty = -1$, and let v_∞ be a "universal" polynomial in x of degree n_∞ with algebraically independent coefficients at this moment. We declare the principal part at infinity of u_1 and u_2 to be equal to v_∞ and $-v_\infty$ respectively. For $\alpha \in S_1 \setminus \{\infty\}$ let v_α be a polynomial in $(x-\alpha)^{-1}$ with terms $(x-\alpha)^{-n_\alpha}, \ldots, (x-\alpha)^{-1}$, also with independent coefficients. We declare the principal part of u_1 and u_2 at α to be equal to

$$v_\alpha + \frac{n/2}{x-\alpha} \qquad \text{and} \qquad -v_\alpha + \frac{n/2}{x-\alpha}$$

respectively. Also let F_1 and F_2 be "general" polynomials in x of degree d_1 and d_2 respectively. We write down

$$u_1 = v_\infty + \sum_{\alpha \in S_1 \setminus \{\infty\}} \left(v_\alpha + \frac{n_\alpha/2}{x-\alpha} \right) - \sum_{\alpha \in S_0} \frac{m_\alpha/2}{x-\alpha} + \frac{F_1'}{F_1}, \tag{3.39}$$

$$u_2 = -v_\infty + \sum_{\alpha \in S_1 \setminus \{\infty\}} \left(-v_\alpha + \frac{n_\alpha/2}{x-\alpha} \right) - \sum_{\alpha \in S_0} \frac{m_\alpha/2}{x-\alpha} + \frac{F_2'}{F_2}. \tag{3.40}$$

These are supposed to be two rational functions satisfying the same Riccati equation. In this way one obtains necessary relations between the coefficients of the chosen general polynomials. The global Galois group G of the corresponding differential equation is supposed to be $\mathbb{G}_\mathrm{m}$. To keep the same Galois group and the same type of equations, it is necessary to require:

(a) The lowest order coefficients of v_α's are non-zero.

(b) For every regular singular point α in $S_1 \setminus \{\infty\}$ we must have $v_\alpha \neq \pm 1/(x-\alpha)$.

(c) If infinity is a regular singular point, then the residues of u_1 and u_2 must be indeed different, and both non-zero.

(d) If all points in S_1 are regular singular, then at least one of them the residue of u_1 is not a rational number. Otherwise the Galois group G would be a finite cyclic group[4] or would contain the additive group $\mathbb{G}_\mathrm{a}$.

(e) The F_i's are not divisible by $x - \alpha$ for any $\alpha \in S_0$. Note that the local exponents of the constructed differential equation at $\alpha \in S_0$ are $-m_\alpha/2$ and $1 + m_\alpha/2$. It follows that if $x - \alpha$ divides F_i then also $(x-\alpha)^{1+m_\alpha}$ divides it. In this case the data for the type changes: $[\alpha]_{m_\alpha}$ becomes $[\alpha]^1$, and some d_i decreases by $m_\alpha + 1$.

[4]The formulas (3.39–3.40) give the form of two global Riccati solutions also in the case when the Galois group G of (3.35) is a finite cyclic group. If the order of G is greater than 2 then these are the only rational solutions of the Riccati equation, and most results in this section hold for differential equations (3.35) with a finite cyclic Galois group of order > 2. Also note that the global Galois group of a second order linear equation $y'' + a\,y' + b\,y = 0$ is $\mathbb{G}_m$ only if the Galois group of its normalized form (3.35) is a subgroup of $\mathbb{G}_m$.

As always, F_i's are determined up to invertible multiples. Their discriminants are invertible if the conditions above are satisfied.

These requirements insure that the rational functions u_1 and u_2 indeed satisfy the given data, and they give us a differential equation $y'' = ry$ with the global Galois group $\mathbb{G}_\mathrm{m}$. Condition *(iv)* is not algebraic, thus the set S_{type} can be represented by an algebraic variety only if the set S_1 contains irregular singular points.

Note that formulas (3.39–3.40) can be used in the Kovacic algorithm ([37]). This algorithm solves second order linear differential equations. At a certain step it looks for rational solutions of the associated Riccati equation. For each pole it computes locally two possible candidates for the principal part of a global Riccati solution. Then it tries to combine these parts to find a global solution. If a Riccati solution u_1 is found, then the global Galois group is a subgroup of the Borel group. At this step one tries to find another Riccati solution u_2 to decide whether the global Galois group is a subgroup of $\mathbb{G}_\mathrm{m}$ or not. We note that a possible u_2 can be written down almost completely following (3.39–3.40). Indeed, one can effectively determine candidates for S_0, these are regular singular points where the local exponents are half-integers. At each singular point one chooses the same principal part as u_1 has, if this singular point is a candidate for S_0, and another one otherwise. It remains to determine F_2. Even if some singular point α was a wrong candidate for S_0, there is still a representation (3.40) for u_2 with $x-\alpha$ dividing F_2.

To obtain more convenient equations between coefficients of v_α's and F_i's one can generalize lemmas 3.2.2 and 3.2.3. The generalized version of equation (3.8) is

$$F_1'F_2 - F_1F_2' + 2\sum_{\alpha\in S_1} v_\alpha\, F_1F_2 = \prod_{\alpha\in S_0}(x-\alpha)^{m_\alpha} \Bigg/ \prod_{\alpha\in S_1\setminus\{\infty\}}(x-\alpha)^{n_\alpha}\,. \tag{3.41}$$

As for equations with two poles, this is convenient when the degrees of the F_i's are small. This formula actually gives the "most universal" family of differential equations with Galois group $\mathbb{G}_\mathrm{m}$: one can choose any F_i's with invertible discriminant and any $\sum_{\alpha\in S_1} v_\alpha$, then computing the left-hand side of (3.41) determines uniquely the set S_0 and m_α's.

The generalized condition (3.12) requires that for every $\alpha \in S_0$ we must have

$$\frac{F_1}{F_2} = \tilde{c}_\alpha \exp\left(-2\int \sum_{\alpha\in S_1} v_\alpha\right) \quad \mathrm{mod} \quad (x-\alpha)^{m_\alpha+1}. \tag{3.42}$$

Here $\tilde{c}_\alpha$ appears because we do not fix a normalization of the F_i's. The integral $\int$ of a power series is chosen to be the primitive power series without the constant term. The relation (3.42) means that F_1/F_2 approximates the exponential function simultaneously at all points of S_0.

The last two "representations" give $\sum_{\alpha\in S_0} m_\alpha$ equations in $\sum_{\alpha\in S_0} n_\alpha + d_1 + d_2 + 1$ variable coefficients (note that v_∞ has $n_\infty + 1$ coefficients). The expected dimension of families of $\mathbb{G}_\mathrm{m}$ differential equations $y = r''y$ with the Galois group $\mathbb{G}_\mathrm{m}$ (or smaller, non-reductive) of the same type is one. The following examples support this expectation. The obtained non-closed curves are even smooth and irreducible.

Example 3.8 Consider the type $([0]_m, [1]^1, [\infty]^0, d_1, d_2)$. We must have $m = d_1 + d_2 + 1$. A priori one can write

$$u_1 = -\frac{m}{2\,x} + \frac{1-c}{2\,(x-1)} - \frac{a}{2} + \frac{F_1'}{F_1}, \quad u_2 = -\frac{m}{2\,x} + \frac{1+c}{2\,(x-1)} + \frac{a}{2} + \frac{F_2'}{F_2},$$

with a, c and coefficients of F_i's undetermined. We normalize F_i's by $F_i(0) = 1$, so Zero is not a root of F_i's automatically. According to (3.42), these u_1 and u_2 satisfy the same Riccati equation if and only if

$$\frac{F_1}{F_2} = (1-x)^c \exp(ax) \mod x^{d_1+d_2+2}.$$

Besides, we must have $a \neq 0$, $c \notin \{0, -1, 1\}$.

First consider the special case $d_2 = 0$. The power series $(1-x)^c \exp(ax)$ is equal to

$$1 + (a-c)x + \frac{a^2 - 2ac + c(c-1)}{2} x^2 + \frac{a^3 - 3a^2c + 3ac(c-1) - c(c-1)(c-2)}{6} x^3 + \ldots$$

Let H_{d_1} denote the (d_1+1)th coefficient in this power series. Then the set of equations of type $([0]_m, [1]^1, [\infty]^0, d_1, 0)$ is represented by the affine curve Spec $C[a, c, a^{-1}c^{-1}(c^2-1)^{-1}]/(H_{d_1})$. If $d_1 = 0$ we get the curve defined by $a = c$, $a \notin \{0, 1, -1\}$. The corresponding differential equation is

$$y'' = \left(\frac{3}{4\,x^2} + \frac{a^2-1}{4\,(x-1)^2} + \frac{a^2x-1}{2x(x-1)} + \frac{a^2}{4} \right) y.$$

If $d_1 \leq 2$ we obtain rational curves parameterized by:

$$d_1 = 1: \qquad a = t(t+1), \qquad c = t^2,$$
$$d_1 = 2: \quad a = \frac{t(t+1)(t+2)}{3\,t+2}, \quad c = \frac{t^3}{3\,t+2}.$$

However, $d_1 = 3$ gives a curve of genus 1. It is isomorphic to $w^2 = t\,(t^2 - 5\,t + 27/4)$, the isomorphism being given by $a - c = 3\,t/(w-2t)$ and $c = 27\,t/2(w-2t)^2$. Computations with Maple show that greater d_1 gives a curve of genus $\lfloor d_1/2 \rfloor \lfloor (d_1-1)/2 \rfloor$ for $d_1 \leq 10$.

To find universal families with $d_2 > 0$, one can introduce F_2 with unknown coefficients and use the expansion of $F_2(1-x)^c \exp(ax)$ at zero. The defining equations are given by coefficients to $d_1+1, \ldots, d_1+d_2+1$ in this power series. The case $d_1 = d_2 = 1$ gives a rational curve parameterized by $a - c = 4\,t/(t^2+3)$ and $c = -16\,t/(t^2+3)^2$. Then $F_1 = (t^2+3) + 2\,(t-1)\,x$, $F_2 = (t^2+3) - 2\,(t+1)\,x$. There are probably no other examples of rational curves in this family.

Example 3.9 Consider the type $S([0]_n, [1]_m, [\infty]^0, d_1, d_2)$. Then $n + m = d_1 + d_2$. According to (3.12) we must have

$$\frac{F_1}{F_2} = \tilde{c}_0 \exp(ax) \mod x^{n+1}, \qquad \frac{F_1}{F_2} = \tilde{c}_1 \exp(a(x-1)) \mod (x-1)^{m+1}.$$

So we are looking for simultaneous Padé approximations at two points.

First suppose that $d_2 = 0$ and $m = 1$. Then we can write down

$$F_1 = \tilde{c}_0 \left(1 + a\,x + \frac{a^2}{2!}\,x^2 + \ldots + \frac{a^n}{n!}\,x^n \right) + c_1\,x^{n+1},$$

where $c_1/\tilde{c}_0$ is determined by the condition $F_1'(1) = a\,F_1$. We can set $c_0 = n+1-a$ and $c_1 = a^{n+1}/n!$. This gives a family of differential equations

$$y'' = \left(\frac{n\,(n+2)}{4\,x^2} + \frac{3}{4\,(x-1)^2} - \frac{n\,(a+1)}{2\,x} + \frac{n-a}{2\,(x-1)} + \frac{a^2}{4} \right) y$$

with Galois group $\mathbb{G}_{\mathrm{m}}$. They are of type $([0]_n, [1]_1, [\infty]^0, n+1, 0)$, unless $a = n+1$ (then $[0]_n$ becomes $[0]^1$, and $d_1 = 0$), or $F_1(1) = 1$ (then $[1]_1$ becomes $[1]^1$), or $a = 0$ (then infinity is not a singular point).

Suppose that $d_2 = 0$ and $m \geq 1$. Then

$$F_1 = \tilde{c}_0 \left(1 + a\,x + \frac{a^2}{2!}\,x^2 + \ldots + \frac{a^n}{n!}\,x^n \right) + c_1\,x^{n+1} + \ldots + c_m\,x^{n+m}.$$

The condition at $x = 1$ is equivalent to $F_1^{(k)}(1) = a\,F_1^{(k-1)}(1)$. This gives m linear equations in c_i's. Its general solution (up to a constant multiple) is

$$\tilde{c}_0 = \sum_{k=0}^{m} (-1)^k \binom{n+m-k}{n} \frac{a^k}{k!}, \quad c_j = \left(\sum_{k=0}^{m-j} (-1)^k \binom{n+m-k}{n} \frac{a^k}{k!} \right) \frac{a^{n+j}}{(n+j)!}.$$

Suppose now that $m = 1$ and $d_2 = 1$. We set $F_2 = 1 + c\,x$ with undetermined coefficient c. Let f denote the transcendental function $F_2 \exp(a\,x)$. Then the polynomial F_1 is equal to the truncated power series $f \mod x^{n+1}$, specifically

$$F_1 = 1 + (a+c)\,x + \frac{a^2 + 2\,c\,a}{2!}\,x^2 + \ldots + \frac{a^n + n\,c\,a^{n-1}}{n!}\,x^n.$$

Besides, we must have:

$$\frac{F_1'(1)}{F_1(1)} = \frac{f'(1)}{f(1)} = \frac{a + c + ac}{1 + c}$$

One can check that

$$(1 + c\,x)F_1' - (a + c + a\,c\,x)F_1 = -\frac{a^{n+1} + (n+1)\,c\,a^n}{n!}\,x^n - \frac{(a^n + n\,c\,a^{n-1})\,a\,c}{n}\,x^{n+1}.$$

From here one derives the equation $a + (n+1)\,c + ac + n\,c^2 = 0$. It defines a rational curve, parameterized by $a = -(t-n)(1+t)/t$ and $c = -(t+1)/t$. Then the polynomial F_1 is equal to

$$\frac{t\,a^n}{n!}x^n + \frac{(t-1)\,a^{n-1}}{(n-1)!}x^{n-1} + \frac{(t-2)\,a^{n-2}}{(n-2)!}x^{n-2} + \ldots + (t-n+1)\,a\,x + (t-n).$$

Example 3.10 Consider differential equations with three regular singular points 0, 1 and ∞. We choose the type $([0]_m, [1]^1, [\infty]^{-1}, d_1, d_2)$. A priori we can write

$$u_1 = -\frac{m}{2\,x} + \frac{1-a}{2\,(1-x)} + \frac{F_1'}{F_1}, \qquad u_2 = -\frac{m}{2\,x} + \frac{1+a}{2\,(1-x)} + \frac{F_2'}{F_2}, \tag{3.43}$$

where a is a parameter. We must have $F_1/F_2 = (1-x)^a \mod x^{m+1}$. Recall that $m = d_1 + d_2$. If F_1/F_2 is the Padé approximation of $(1-x)^a$ of degree (d_1, d_2), then u_1 and u_2 satisfy the same Riccati equation. If these two rational functions are distinct, then the Galois group of the corresponding linear differential equation must be $\mathbb{G}_\mathrm{m}$ or a proper subgroup of it.

The differential equation can be obtained from (3.43) by computing just first local terms of $u_i' + u_i^2$:

$$y'' = \left(\frac{m\,(m+2)}{4\,x^2} + \frac{a^2-1}{4\,(x-1)^2} + \frac{a\,(d_2-d_1) - 2\,d_1\,d_2 - m}{2\,x\,(x-1)} \right) y. \tag{3.44}$$

It is the normalization of the standard hypergeometric equation (see [4])

$$x\,(x-1)\,y'' + ((1-a-m)\,x + m)\;y' + d_1\,(a+d_2)\,y = 0. \tag{3.45}$$

This hypergeometric equation is also the equation for F_1. The three "standard" constants for it are $-d_1, -a-d_2, -m$. The corresponding hypergeometric series is

$$\sum_{k=0}^{\infty} \frac{(-d_1)_k\,(-a-d_2)_k}{(-d_1-d_2)_k\,k!}\,x^k, \qquad (z)_k = z\,(z+1)\ldots(z+k-1).$$

Since $-d_1 - d_2 = -m$ is a negative integer, this series is not well-defined. However it does give us the polynomial of degree d_1 satisfying the hypergeometric equation. We can write F_1, and similarly F_2, as

$$F_1 = \sum_{k=0}^{d_1} \frac{(-d_1)_k\,(-a-d_2)_k}{(-d_1-d_2)_k\,k!}\,x^k, \qquad F_2 = \sum_{k=0}^{d_2} \frac{(-d_2)_k\,(a-d_1)_k}{(-d_1-d_2)_k\,k!}\,x^k. \tag{3.46}$$

If $a \notin \mathbb{Q}$ then the two rational solutions u_1 and u_2 of the Riccati equation are distinct. Then the Galois group of (3.44) is clearly $\mathbb{G}_\mathrm{m}$, and one can write a general solution of it as

$$y = C_1\,x^{-\frac{m}{2}}\,(x-1)^{\frac{1-a}{2}}\,F_1 + C_2\,x^{-\frac{m}{2}}\,(x-1)^{\frac{1+a}{2}}\,F_2. \tag{3.47}$$

For the values of a in $\mathbb{Q}$ we have the following possibilities for the Galois group of (3.44).
(a) It is a finite cyclic group of order ≥ 2, if $a \in \mathbb{Q} \setminus \mathbb{Z}$.
(b) It is trivial or $\mathbb{Z}/2\mathbb{Z}$, if $a \in \mathbb{Z}$, and $a < -d_2$ or $a > d_1$.
(c) It is the additive group $\mathbb{G}_\mathrm{a}$ or $\mathbb{G}_\mathrm{a} \times \mathbb{Z}/2\mathbb{Z}$, if $a \in \mathbb{Z}$ and $-d_2 \leq a \leq d_1$.

In the case (a) the rational solutions (3.43) of the Riccati equation give us a basis of algebraic solutions of the same form as in (3.47) for the differential equation (3.44).

If $a \in \mathbb{Z}$ but $a < -d_2$ or $a > d_1$ then the two rational Riccati solutions (3.43) are still different, because they have different residues at infinity. Then (3.47) gives a basis of rational (or "almost rational") solutions of (3.44). Assume that $a > 0$. Then using $F_1/F_2 = (1-x)^a \mod x^{m+1}$ and knowing the two local exponents at infinity one finds the solution $x^{m/2+1}\,(x-1)^{(1-a)/2}\,F_3$ of (3.44), with F_3 of degree $a - d_1 - 1$. If the degree of F_3 is small, we can replace one of the basis solutions in (3.47) by this "easier" solution.

In the case (c) either the actual degree of F_1 in (3.46) is $d_2 + a$ if $-d_2 \le a \le d_1 - d_2$, or $\deg F_2 = d_1 - a$ if $d_1 - d_2 \le a \le d_1$. It follows that the two solutions in (3.47) have the same order at infinity. Moreover, this order is the maximum of the local exponents at infinity, thus the two solutions are equal up to a constant multiple. By writing down the hypergeometric series at the infinity or at $x = 1$ one concludes that there are no local solutions with the smaller local exponent at these points, hence the local Galois group at these points is $\mathbb{G}_a$. However, the local Galois group at zero is trivial or $\mathbb{Z}/2\mathbb{Z}$, because here we have a local solution F_1 or F_2 with the smaller local exponent. This proves that the global Galois group is $\mathbb{G}_a$ or $\mathbb{G}_a \times \mathbb{Z}/2\mathbb{Z}$. Suppose that $a \ge d_1 - d_2$. A few considered examples suggest that a basis of solutions of (3.44) in $C(x, \log(1-x))$ is

$$y = C_1\, x^{-\frac{m}{2}}\,(x-1)^{\frac{1+a}{2}}\,F_2 + C_2\, x^{-\frac{m}{2}}\,\left((x-1)^{\frac{1+a}{2}}\,F_2\,\log(1-x) - (x-1)^{\frac{1-a}{2}}\,\tilde{F}_1\right),$$

with $\deg F_2 = d_1 - a$, and $\tilde{F}_1$ being a polynomial of degree m equal to the truncated power series $(x-1)^a\,F_2\,\log(1-x) \mod x^{m+1}$; here $\log(1-x) = x + x^2/2 + x^3/3 + \ldots$ in $C((x))$.

One can compare this example with the classification in [10], or theorem 4.2.1 in [32] which classifies reducible representations of the free group with two generators, which give the possible monodromy groups of the hypergeometric differential equations.